12. Intro to CFT

Read: Simmons-Duffin 1602.07982
Sections 3, 4, 5, 6

First some basic terminology :

Symmetries
- Act on fixed theory
- relate states → other states

Scale invariance

$x \to \lambda x$

no intrinsic length scale

Conformal invariance

diffs $x \to x'$ such that
$ds^2 \to \Omega(x)^2 ds^2$

Not a "symmetry"
- Acts on bg. fields
- relates observables in one theory on M_1 to a different theory on M_2

Weyl invariance

$ds^2 \to \Omega^2(x) ds^2$, NOT necessarily a diff.

It's mostly true that

$$\boxed{\text{Scale}} \Rightarrow \boxed{\text{conformal in flat space}} \Rightarrow \boxed{\text{Weyl up to anomalies}}$$

The basic reason is

Locality

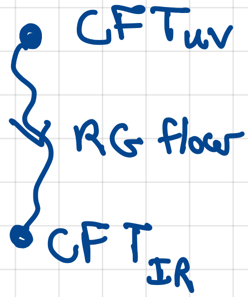
That is, for a QFT to be scale invariant, it must be locally scale invariant, and this (almost) implies the others.

I'll give a more detailed statement soon.

CFT = Building blocks for QFT

Almost any QFT that is well-defined @ all scales is conformal in UV and IR:

QFT =



Thus QFT is (almost) the problem of understanding CFTs and their deformations.

Caveat: this paradigm misses some physically interesting theories like QED_4 and ϕ^4 (Landau Pole)

Classically scale invariant FT's

Easy: anything w/ no dimensionful parameters.

* $\int d^4x \left[(\partial\phi)^2 \right] \quad = \quad \text{CFT}$

* Massless QED_4 $\qquad \alpha \cong \frac{1}{137}$ $\qquad$ No scale classically.

But β-fn is non-zero

$$\alpha \Big|_{90\ \text{GeV}} \cong \frac{1}{127}$$

Not a scale-invariant QFT.

* $\int d^4x \left[(\partial\phi)^2 + g\phi^4 \right] \quad = \quad$ not a CFT because $\beta_g \neq 0$

But classically,

$$\partial^2\phi = g\phi^3$$

given sol'n $\phi_1(x)$, find another sol'n

$$\phi_2(x) = \lambda^\Delta \phi_1(\lambda x) \quad \text{where } \Delta = 1 = \text{mass dimension of } \phi$$

$$x \to x' = \lambda x$$

$$\phi(x) \to \phi'(x') = \lambda^{-\Delta}\phi(x)$$

A few CFTs

* $\int (\partial\phi)^2$, $\int \bar{\psi} \not{\partial} \psi$

* $\int d^2x \; G_{ij}(\phi) \, \partial_\mu \phi^i \partial^\mu \phi^j + \bar{\psi}^i \not{\partial} \psi_i$

* 4d $\mathcal{N}=4$ SYM

* $\int d^3x \left[(\partial\vec{\phi})^2 + m^2 \vec{\phi}^2 + g \vec{\phi}^4 \right]$ in IR

  (critical Ising)

  Note that $\mathcal{L}$ is not a great tool to study this theory. It's strongly interacting. The most precise calculations (bootstrap) do not use $\mathcal{L}$ whatsoever.

* 2d minimal models

etc.

CFT Basics

recall generators:

$$\delta_\xi g_{\mu\nu} = 2\sigma(x)\, g_{\mu\nu}$$
$$= 2\nabla_{(\mu}\xi_{\nu)}$$

$\Rightarrow$ Poincaré

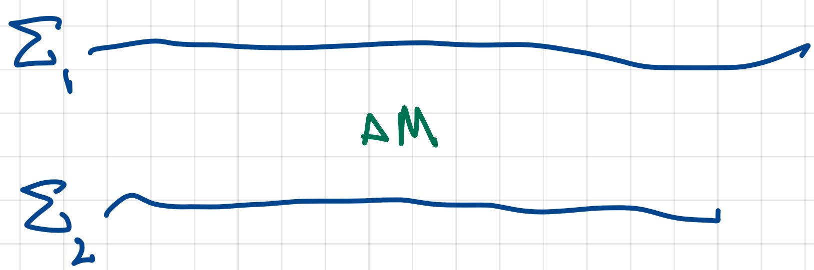
$+ \; k_\mu$

$+ \; \xi = x^\mu \partial_\mu$

each generator $\rightsquigarrow$ conserved charge

$$Q[\xi] = \int_{\text{space}} d\Sigma^\mu \, T_{\mu\nu} \, \xi^\nu$$

"conserved" means



$$Q[\xi]_{\Sigma_1} = Q[\xi]_{\Sigma_2}$$

$$0 = \Delta Q = \int_{\Delta M} d^dx \sqrt{g} \; T^{\mu\nu} \underbrace{\nabla_\mu \xi_\nu}$$

(used $\nabla_\mu T^{\mu\nu} = 0$ and I.B.P.)

$$\nabla_{(\mu}\xi_{\nu)} = \frac{1}{2}\delta_\xi g_{\mu\nu}$$
$$= \sigma(x)\, g_{\mu\nu}$$

$$= \int_{\Delta M} d^d x \sqrt{g}\, T^{\mu}_{\ \mu}(x)\, \sigma(x)$$

$$= 0$$

Dilatation: $\xi = x^\mu \partial_\mu , \quad \delta_\xi g_{\mu\nu} = g_{\mu\nu} , \quad \sigma = \frac{1}{2}$

$$D := Q[x^\mu \partial_\mu]$$

$$\Delta D = \frac{1}{2} \int_{\Delta M} d^d x \sqrt{g}\, T^\mu_{\ \mu}(x)$$

$$= 0$$

This almost(*) implies

$$T_\mu^{\ \mu}(x) = 0$$

In this case, all the other conformal charges are automatically conserved.

$$\boxed{T_\mu^{\ \mu}(x) = 0 \iff \text{Conformal invariance}}$$

Fine print

- CFT in curved space is often NOT quite conformal (anomaly)

  but still called a "CFT".

  ex: $S^{d-1}_R \times$ time

  $E_i \propto 1/R$

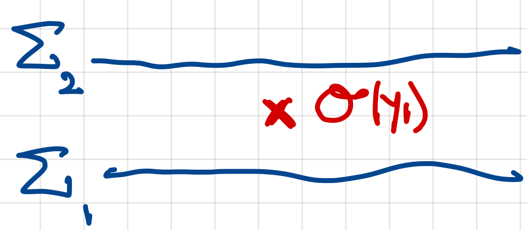
- "operator equation": really means

$$\langle T^\mu_\mu(x)\, \mathcal{O}(y_1)\, \mathcal{O}(y_2) \cdots \rangle = 0 + \underbrace{\sum_i \delta(x-y_i) * \text{stuff}}_{\text{contact terms}}$$

Similarly for

$$\nabla_\mu T^{\mu\nu} = 0$$

ex:



$$Q[\xi]_{\Sigma_1} \neq Q[\xi]_{\Sigma_2}$$ due to contact terms in $T(x)\mathcal{O}(y)$

Aside:
Scale vs. Conformal

Why the "almost"?

If $T^\mu_{\ \mu} = \nabla_\mu V^\mu$, then the "improved" dilatation current

$$j_D^\mu = T^\mu_{\ \nu} x^\nu - V^\mu$$

is conserved, and theory is scale inv. but not CFT. (unless $V^\mu = \nabla_\nu L^{\mu\nu}$)

V^μ = Virial current
= "internal part" of scale transformation

$[T_{\mu\nu}] = d$, so would require

$[V^\mu] = d-1$ and $\nabla_\mu V^\mu \neq 0$

Typically no such operator;

Proven not to exist in $d=2$ $(d=4)$

perturbatively or with some extra assumptions

"Scale $\Rightarrow$ conformal"

Ex. 3d Maxwell is scale inv, not CFT

$V^\mu \propto A_\nu F^{\mu\nu}$

j_D^μ is Not gauge invariant!

but D is.

(Not clear if we should call this a scale symmetry...)

Possibly: scale current $\Rightarrow$ CFT in all d.

Operators

(scalar $\mathcal{O}(x)$) $\mathcal{O}(x)$ is called "primary" if

$$[D, \mathcal{O}(0)] = \Delta\, \mathcal{O}(0)$$

scaling dimension

$$[K_\mu, \mathcal{O}(0)] = 0$$

finite version:

under $x \to \lambda x$,

$$\mathcal{O}(x) \to \lambda^\Delta \mathcal{O}(\lambda x) \qquad [\text{cf. } \phi^4 \text{ example}]$$

for general conformal transformations,

$x \to x'$ with

$$g'_{\mu\nu}(x) = \Omega(x)^{-2} g_{\mu\nu}(x)$$

(Let's check this is really Ω^{-2}, not Ω^2:

For $x' = \lambda x$, $ds^2 = dx^2 = \frac{1}{\lambda^2} dx'^2$, so $g' = \frac{1}{\lambda^2} g$ ✓)

$$\mathcal{O}(x) \to \Omega(x)^\Delta \mathcal{O}(x')$$

Caveat: ds^2 is fixed

Correlators (of primaries)

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \cdots \rangle_{ds^2\,(g_{\mu\nu})}$$

$$= \Omega(x_1)^{\Delta_1}\, \Omega(x_2)^{\Delta_2} \cdots \langle \mathcal{O}_1(x_1')\, \mathcal{O}_2(x_2') \cdots \rangle_{ds^2\,(g'_{\mu\nu})}$$

(Note M.f. is unchanged!)

Conformal transf. fix:

usually pick basis so $C_{12} = \delta_{12}$

$$\langle \mathcal{O}_1(x)\,\mathcal{O}_2(y)\rangle = \begin{cases} \dfrac{C_{12}}{|x-y|^{2\Delta}} & \text{for } \Delta_1 = \Delta_2 \\ 0 & \text{otherwise} \end{cases}$$

check: $\langle \mathcal{O}(\lambda x)\,\mathcal{O}(\lambda y)\rangle = \lambda^{-2\Delta}\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle$ (metric $dx_a dx_a = \frac{1}{\lambda^2} dx'_a dx'_a$)

and

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\,\mathcal{O}_3(x_3)\rangle$$

"OPE coefficient"

$$= \frac{C_{123}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}\,|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}\,|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}$$

"CFT in $\mathbb{R}^d$ = list of scaling dims Δ_i + OPE coeff C_{ijk}"

(+ non-local stuff in $d > 2$?)

Recap end of prev. lecture:

$$g'_{\mu\nu}(x') = \Omega^{-2}(x)\, g_{\mu\nu}(x')$$

inf'ly,

$$x \to x' = x - \varepsilon$$

Action on operators:

$$\mathcal{O}(x) \to \mathcal{O}'(x) = e^{Q[\varepsilon]}\, \mathcal{O}(x)\, e^{-Q[\varepsilon]}$$

Scalar primary:

$$\mathcal{O}'(x) = \Omega(x')^{\Delta}\, \mathcal{O}(x')$$

write to save:

$$\langle \mathcal{O}_1(x_1) \cdots \rangle = \Omega(x_1')^{\Delta_1} \cdots \langle \mathcal{O}_1(x_1') \cdots \rangle$$

"descendant" operators

$$[P_{\mu_1}, \cdots [P_{\mu_n}, \mathcal{O}(0)]] = \partial_{\mu_1} \cdots \partial_{\mu_n} \mathcal{O}(0)$$

momentum ops

have dimension $\Delta + n$.

All operators can be organized into lowest-wt. reps with primary $\mathcal{O}(0)$ + descendants

[To see this, act w/ K_μ. using conformal algebra,

$$D K_\mu \mathcal{O}(0) = (\Delta - 1) K_\mu \mathcal{O}$$

so K_μ lowers Δ. Unitarity requires $\Delta > 0$, so this must terminate at a primary]

Weyl

we'll see why conformal $\Rightarrow$ Weyl later ...

In a Weyl-invariant theory, for any $\Omega^2(x)$,

$$\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\cdots\rangle_g$$

$$= \Omega(x_1)^{\Delta_1}\,\Omega(x_2)^{\Delta_2}\cdots\langle \mathcal{O}_1(x_1)\,\mathcal{O}_2(x_2)\cdots\rangle_{\Omega^2 g}$$

If $\Omega^2 g_{\mu\nu}$ is the same mf. as $g_{\mu\nu}$, then can reinterpret as a conformal transf.

Ex:

The conformal transf. $x' = \lambda x \Rightarrow$

$$\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle_{\delta_{\mu\nu}}$$

$$= \lambda^{2\Delta}\langle \mathcal{O}(\lambda x)\,\mathcal{O}(\lambda y)\rangle_{dx^2} \qquad (x' = \lambda x \text{ conf. transf.})$$

$$= \lambda^{2\Delta}\langle \mathcal{O}(x)\,\mathcal{O}(y)\rangle_{\lambda^2 dx^2}$$

Weyl law for $\Omega = \lambda$

CFTs are Weyl inv. (up to anomaly) b/c in curved space,

$$T^\mu_\mu(x) = A(x)$$

operator $\uparrow$ $\uparrow$ c-number built from bg fields $R, R_{\mu\nu}, \ldots$

(We'll derive this soon in 2d.)

Therefore

under $g_{\mu\nu} \to g_{\mu\nu}(1+2\sigma(x))$,

$$\delta \log Z[g] = \int d^dx \sqrt{g}\, \langle T^\mu_\mu(x) \rangle\, \sigma(x)$$

$$\delta \langle \mathcal{O}(x_1) \cdots \rangle$$

$$= \int d^dx \sqrt{g}\, \langle \underbrace{T^\mu_\mu(x)\, \mathcal{O}(x_1)}_{\text{contact term } \Delta \mathcal{O}(x_1)\delta(x-x_1)} \cdots \rangle_g\, \sigma(x)$$

(homework)

$$= \Delta\, \sigma(x_1) \langle \mathcal{O}(x_1) \cdots \rangle + \text{other contact terms}$$

$$= \text{inf'l Weyl law}$$

(Optional) Conformal $\Rightarrow$ Weyl in 2d

In flat space,

$$T_\mu^{\ \mu} = 0$$

$\Rightarrow$ In curved space $g_{\mu\nu}$, most general possibility by dimensional analysis:

"central charge" (defn.)

$$T_\mu^{\ \mu}(x) = -\frac{c}{24\pi} R + R\mathcal{O} + \cancel{\mathcal{O}(\nabla^4)}$$

$[T] = d = 2$

$R[g]$ c-number $R \sim \nabla^2 g$ so $[R] = 2$

$\mathcal{O}$ = dim-0 scalar operator

~~$R^2\tilde{\mathcal{O}}$, $R\,\Box\,\tilde{\mathcal{O}}$, etc~~

~~$[\tilde{\mathcal{O}}] = -2$ impossible~~

Can a CFT have dim-0 operator?

Yes, sort of: $\int \partial X \partial X$, $[X] = 0$

We will now rule out such an operator in the trace.

Claim: $R\mathcal{O}$ cannot appear in $T_\mu^{\ \mu}$

Intuition: this would relate correlators of $\mathcal{O}$ to metric deformations in an impossible way.

Consider

$$Z[g,J] = \left\langle e^{\int d^dx\, J(x)\mathcal{O}(x)} \right\rangle_g$$

Vary:

$$\delta g_{\mu\nu} \equiv 2\sigma g_{\mu\nu}$$

$$\delta J \equiv \sigma R$$

Then

$$\delta \log Z = \int d^2x \left(\delta g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} + \delta J \frac{\delta}{\delta J} \right) \log Z$$

$$= \int d^2x \sqrt{g} \left(-\sigma(x) \langle T_\mu^{\ \mu}(x) \rangle_{g,J} + \sigma(x) R(x) \langle \mathcal{O}(x) \rangle_{g,J} \right)$$

$$= \frac{c}{24\pi} \int d^2x \sqrt{g}\, \sigma(x) R(x)$$

Now we want to act again.

In conformal gauge,

$$ds^2 = e^{2\sigma} \delta_{\mu\nu} dx^\mu dx^\nu$$

$$\sqrt{g} R = -2 \partial_\mu \partial^\mu \sigma$$

$$\delta_{\sigma_1} \delta_{\sigma_2} \log Z = -\frac{c}{12\pi} \int d^2x \; \sigma_1 \partial_\mu \partial^\mu \sigma_2$$

Thus

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] \log Z = 0 \qquad \text{(I.B.P.)}$$

On b.g. fields,

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] g_{\mu\nu} = 0$$

$$[\delta_{\sigma_1}, \delta_{\sigma_2}] J = 2(\sigma_1 \partial^2 \sigma_2 - \sigma_2 \partial^2 \sigma_1)$$

$$= \text{any function you want!}$$

Therefore

$$\frac{\delta}{\delta J} \log Z[g, J] = 0$$

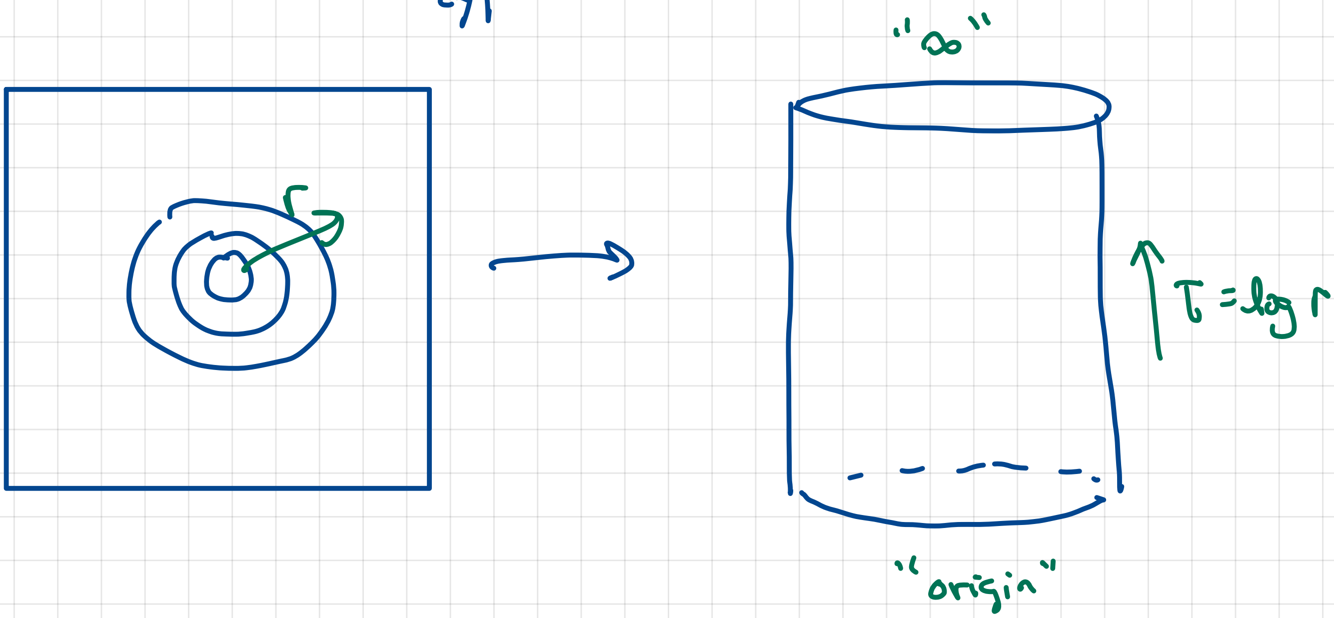
Therefore $\Theta = 0$ ✓

$$\boxed{\text{Conclude: In 2d, } T^\mu_\mu = \frac{-c}{24\pi} R}$$

Weyl mapping to Cylinder

$$\mathbb{R}^d \xrightarrow{\text{Weyl}} \mathbb{R} \times S^{d-1}$$

$$\begin{aligned}
ds^2_{pl} &= dr^2 + r^2 d\theta^2_{d-1} \\
&= r^2 \left(\frac{dr^2}{r^2} + d\theta^2_{d-1} \right) \\
&= e^{2\tau} \left(d\tau^2 + d\theta^2_{d-1} \right) \\
&= e^{2\tau} \, ds^2_{cyl}
\end{aligned}$$



Dilatations $r \to \lambda r$ $\Longrightarrow$ $\tau \to \tau + \log \lambda$

time translations

$D \Rightarrow H$ (+ anomaly = $+\frac{c}{12}$ in 2d)

$\Delta_i \Rightarrow E_i$ (+ $\frac{c}{12}$ in 2d

2-pt. fn

$$\langle \mathcal{O}(\tau_1, n_1)\, \mathcal{O}(\tau_2, n_2) \rangle_{pl}$$ $\leftarrow$ $\mathbb{R}^d$ correlator in polar coords

$$= \langle \mathcal{O}(x_1 = e^{\tau_1} n_1)\, \mathcal{O}(x_2) \rangle_{pl}$$

$$= |x_1 - x_2|^{-2\Delta}$$

$$= \left| e^{2\tau_1} + e^{2\tau_2} - 2\, e^{\tau_1 + \tau_2}\, n_1 \cdot n_2 \right|^{-\Delta}$$

Weyl:

$$ds^2_{pl} = \Omega^2\, ds^2_{cyl}\,, \qquad \Omega = e^{\tau}$$

$\Rightarrow$

$$\langle \mathcal{O}(\tau_1, n_1)\, \mathcal{O}(\tau_2, n_2) \rangle_{cyl}$$

$$= \Omega(x_1)^{\Delta}\, \Omega(x_2)^{\Delta} \langle \mathcal{O}(\tau_1, n_1)\, \mathcal{O}(\tau_2, n_2) \rangle_{pl}$$

$$= e^{\Delta(\tau_1 + \tau_2)} \left| e^{2\tau_1} + e^{2\tau_2} - 2\, e^{\tau_1 + \tau_2}\, n_1 \cdot n_2 \right|^{-\Delta}$$

for $n_1 = n_2$:

$$= \left| 2 \sinh\left(\tfrac{1}{2} (\tau_1 - \tau_2) \right) \right|^{-2\Delta}$$

observe:

* as $\tau_1 \to \tau_2$, $\langle \mathcal{O}\mathcal{O} \rangle_{cyl} \sim |\tau_1 - \tau_2|^{-2\Delta}$

$$= |\text{proper dist.}|^{-2\Delta}$$

(always!)

* for $\tau_2 \gg \tau_1$, decays exponentially.

This holds for all correlators on cylinder (may require some smearing)

gapped.

Shortcut:

Define $\quad \mathcal{O}_{cyl}(\tau, n) = e^{\tau\Delta}\, \mathcal{O}(x = e^{\tau} n)$

then

$$\langle \mathcal{O}_{cyl}(\tau_1, n_1)\, \mathcal{O}_{cyl}(\tau_2, n_2) \rangle$$

$$= e^{(\tau_1+\tau_2)\Delta} \langle \mathcal{O}(x_1)\, \mathcal{O}(x_2) \rangle$$

by substitution.

I find this very confusing: $\mathcal{O}_{cyl}$ is an operator in the flat-space CFT that "mimics" the physics in curved space.

But this is very standard.

Worse, people don't write "cyl", we're supposed to know from its arguments.

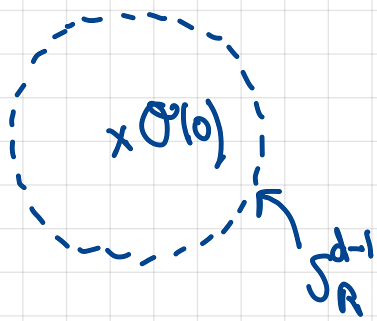
State - Operator Correspondence

In any QFT,

local operators $\rightarrow$ states on S^{d-1}_R

via

$$|\mathcal{O}\rangle = \text{P.I. on ball } |x| \leq R$$

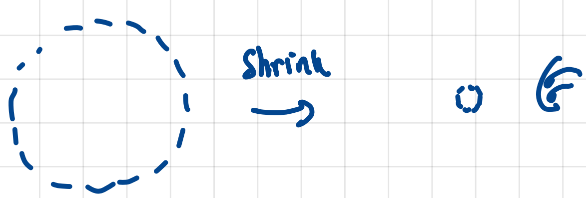


This $|\mathcal{O}\rangle \in \mathcal{H}_{S^{d-1}_R}$

In CFT,

* this $\mathcal{H}$ is indep. of R; set $R=1$

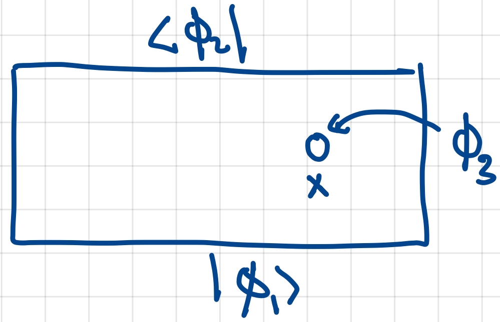
* any state $\rightarrow$ local operator:

$|\psi\rangle$ on 

$\Leftarrow$ insert this into path integral defines a local op.

ex: $\langle \phi_2 | \psi(x) | \phi_1 \rangle$

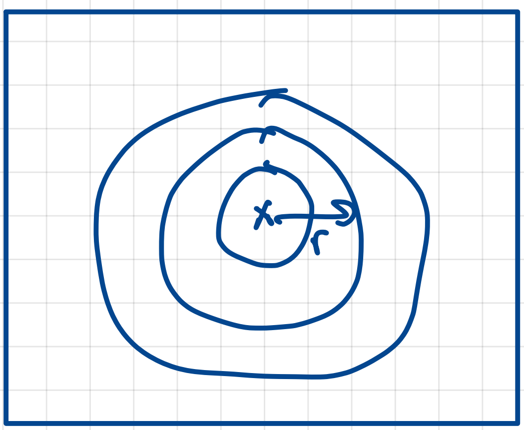
$$= \int D\phi_3 \; \underbrace{\psi[\phi_3]}_{\text{wavefn.}}$$



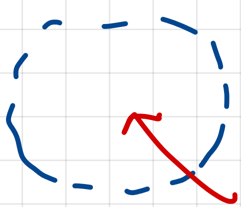
$\Rightarrow$ "1-1 State-Operator correspondence"

"Radial quantization"

r = "Euclidean time"

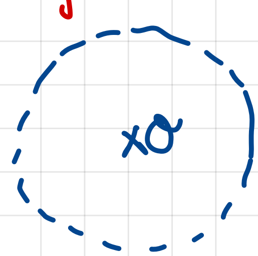


$|0\rangle \equiv$



regular (no operator) @ origin

$\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle =$



$\langle 0 | \equiv$ "nothing @ ∞"

$\langle \mathcal{O} | = \langle 0 | \mathcal{O}(\infty)$

where $\mathcal{O}(\infty) \equiv \lim\limits_{x \to \infty} x^{2\Delta} \mathcal{O}(x)$

Ex:

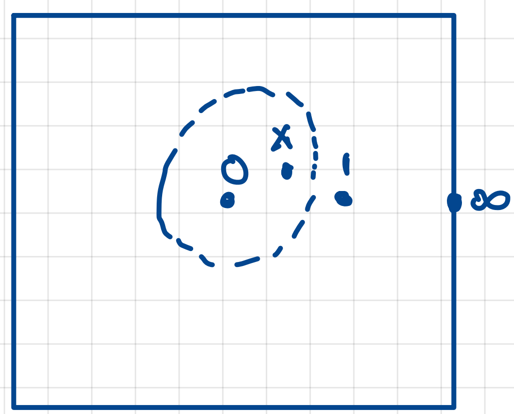
$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \lim_{x \to \infty} x^{2\Delta_1} \langle \mathcal{O}_1(x) \mathcal{O}_2(0) \rangle$$

assume $\Delta_1 = \Delta_2$

$$= \lim x^{2\Delta} x^{-2\Delta} C_{12}$$

$$= C_{12}$$

Ex:



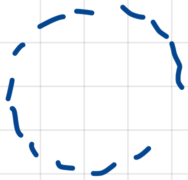
$$\langle \mathcal{O}(\infty) \mathcal{O}(1) \mathcal{O}(x) \mathcal{O}(0) \rangle$$

$$= \langle \mathcal{O} | \mathcal{O}(1) \mathcal{O}(x) | \mathcal{O} \rangle$$

$$= \sum_n \langle \mathcal{O} | \mathcal{O}(1) | n \rangle \langle n | \mathcal{O}(x) | \mathcal{O} \rangle$$

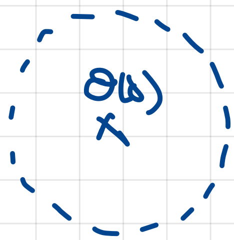
The Cylinder

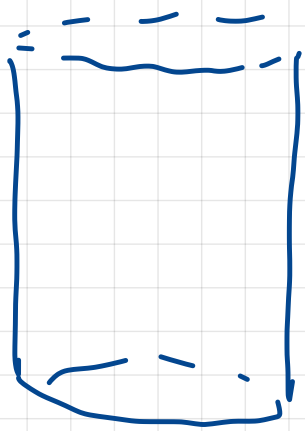
Under the Weyl mapping we described earlier,

$|0\rangle_{\text{radial quant}} =$  $\Longrightarrow$ $|0\rangle =$  $\tau = -\infty$

$=$ Path integral w/ "regularity" at $\tau = -\infty$

$=$ usual vacuum state!

$|\mathcal{O}\rangle =$ 



$|\mathcal{O}\rangle$

energy eigenstate